Derivatives of higher order:

- Let $E \subset \mathbb{R}^n$ be open & $f: E \to \mathbb{R}$ with partial $D_j f$ which are themselves differentiable. Then, the second order partials are defined by

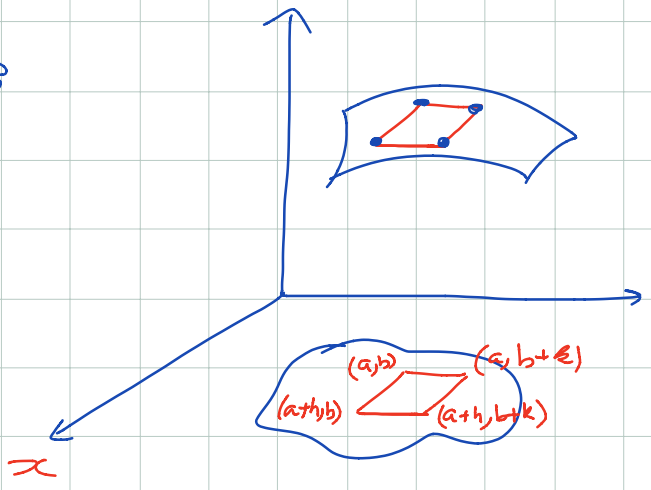
$$D_{ij} f = D_i D_j f \quad \text{where } i,j = 1, \dots, n$$

We say that $f \in C''(E)$ if each of the $D_{ij} f$ is continuous

- $f: \underset{\text{open}}{E} \to \mathbb{R}^m$ is C'' if each of its m component functions is continuous.

Remark: $D_{ij} f \neq D_{ji} f$ is possible unless both are continuous.

MVT:



Theorem (MVT): Let $f: E \subset \mathbb{R}^2 \to \mathbb{R}$ and suppose E open

$D_1 f$ & $D_{21} f$ exist at every point in E.

Suppose further that $Q \subset E$ is a closed rectangle with sides $\parallel$ to the coordinate axes. Let (a,b) & $(a+h, b+k)$ be its opp. vertices and set

$$\Delta(f,Q) = f(a+h,b+k) - f(a,b+k) - f(a+h,b) + f(a,b)$$

Then $\exists (x,y) \in \text{interior}(Q)$ s.t.

$$\Delta(f,Q) = hk\,(D_{21}f)(x,y).$$

Proof: By MVT (in 1 variable) $\exists\, \xi \in (a, a+h)$

s.t. $\overbrace{u(a+h) - u(a)}^{\Delta(f,Q)} = u'(\xi)\,h$

where $u(t) = f(t, b+k) - f(t,b)$.

So $\Delta(f,Q) = h\left(D_1 f(\xi, b+k) - D_1 f(\xi, b)\right)$

$\Rightarrow$ by MVT $\Delta(f,Q) = hk\,(D_{21} f(\xi, \nu))$ for some $\nu \in (b, b+k)$

Theorem: Let $E \subset \mathbb{R}^2$ be open & $f: E \to \mathbb{R}$ be such that $D_1 f$, $D_2 f$ & $D_{21} f$ all exist at every pt in E & $D_{21} f$ is cont's at $(a,b) \in E$.

Then: $D_{12} f$ exists at (a,b) &

$$(D_{12} f)(a,b) = (D_{21} f)(a,b).$$

In Particular: $D_{21} f = D_{12} f$ if $f \in C''(E)$.

Proof: Want to show $(D_{12} f)(a,b)$ exists & $= (D_{21} f)_{(a,b)}$

i.e., we want to show that

$$\lim_{h \to 0} \underbrace{\frac{D_2 f(a+h,b) - D_2 f(a,b)}{h}}_{\text{Defn of } D_{12} \text{ if it exists}} - D_{21} f(a,b) = 0$$

Fix $\varepsilon > 0$, $\exists\, h, k$ small enough so that

$$\left| (D_{21} f)(a,b) - (D_{21} f)(x,y) \right| < \varepsilon \quad \forall x, y \in Q := [a, a+h] \times [b, b+k]$$

$D_{21} f(\xi, \nu)$ for some $(\xi, \nu) \in Q$

$$\Rightarrow \left| \frac{\Delta(f, Q)}{hk} - (D_{21} f)(a,b) \right| < \varepsilon \qquad \circledast$$

(By the previous (MVT) theorem)

But $\Delta(f,Q) = f(a+h,b+k) - f(a+h,b) - f(a,b+k) - f(a,b)$

$\Rightarrow \lim_{k\to 0} \frac{\Delta(f,Q)}{k} = D_2 f(a+h,b) - D_2 f(a,b)$

Fixing h in $\circledast$ & letting $k \to 0$ yields

$$\left| \frac{D_2 f(a+h,b) - D_2 f(a,b)}{h} - D_{21} f(a,b) \right| < \varepsilon$$

$\Rightarrow D_{12} f(a,b) = D_{21} f(a,b)$.



Differentiation of Integrals

Goal: Derive conditions so that

$$\frac{d}{dt} \int_a^b \varphi(x,t)\,dx = \int_a^b \frac{\partial \varphi}{\partial t}(x,t)\,dx$$

Remark: not always true!

Theorem: Suppose $\varphi : [a,b] \times [c,d] \to \mathbb{R}$

& α is an increasing $\uparrow$ on $[a,b]$
where $\varphi_t(x) = \varphi(x,t) \in \mathcal{R}(\alpha)$ $\forall t \in [c,d]$.

Suppose that $\forall s \in (c,d)$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.:

$$|(D_2\varphi)(x,t) - D_2\varphi(x,s)| < \varepsilon \qquad (*)$$

holds $\forall x \in [a,b]$ & $t \in (s-\delta, s+\delta)$.

Define $f(t) = \int_a^b \varphi(x,t)\, d\alpha(x)$ $(t \in [c,d])$

Then $(D_2\varphi)(\cdot, s) \in \mathcal{R}(\alpha)$, and $f'(s)$ exists and satisfies

$$f'(s) = \int_a^b (D_2\varphi)(x,s)\, d\alpha(x).$$

Proof:

Let $\psi(x,t) = \dfrac{\varphi(x,t) - \varphi(x,s)}{t-s}$ where $|t-s| < \delta$

By the standard MVT, $\exists u \in (s,t)$ s.t.

$$\psi(x,t) = (D_2\varphi)(x,u)$$

Then, by (**) $|\underbrace{\psi(x,t)}_{D_2\varphi(x,u)} - D_2\varphi(x,s)| < \varepsilon \quad \forall x \in [a,b]$

$\Rightarrow \psi(\cdot,t) \xrightarrow[t\to s]{} D_2 f(\cdot,s)$ uniformly on $[a,b]$

But $\dfrac{f(t)-f(s)}{t-s} = \displaystyle\int_a^b \underbrace{\frac{\varphi(x,t)-\varphi(x,s)}{t-s}}_{\psi(x,t)=\psi_t(x)\in \mathcal{R}(\alpha)} d\alpha(x)$

$\Rightarrow$ taking limits $f'(s) = \displaystyle\int_a^b (D_2\varphi)(x,s)\, d\alpha(x)$ ∎

(α is monotonically increasing & $g_n \in \text{Range}(\alpha)$)

$\displaystyle\int_a^b \lim_{n\to\infty} g_n \, d\alpha = \lim_{n\to\infty} \int_a^b g \, d\alpha$

—X—

Let's finish our discussion of Chapter 9 with an example of the Implicit Ft'n theorem

Let $f: \mathbb{R}^5 \to \mathbb{R}^2$

be given by

$$\begin{cases} f_1(x_1,x_2,x_3,x_4,x_5) = 2e^{x_1} + x_2x_3 - 4x_4 + 3 \\ f_2(x_1,x_2,x_3,x_4,x_5) = x_2\cos x_1 - 6x_1 + 2x_3 - x_5 \end{cases}$$

and note that $F(0,1,3,2,7) = 0$.

Moreover $[F'(0,1,3,2,7)] = [A] = \underbrace{\begin{bmatrix} 2 & 3 & 1 \\ -6 & 1 & 2 \end{bmatrix}}_{A_x} \underbrace{\begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}}_{A_y}$

Note that A_x is invertible $\Rightarrow$ by the implicit fct theorem $\exists\, g: \mathbb{R}^3 \to \mathbb{R}^2$, continuously diff. in the neighborhood of $(3,2,7)$

with $g(3,2,7) = (0,1)$ & $F(g(y), y) = 0$

Moreover $[g'(3,2,7)] = -A_x^{-1} A_y$

$$= \frac{-1}{20}\begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1/4 & 1/5 & -3/20 \\ -1/2 & 6/5 & 1/10 \end{bmatrix}}_{\substack{\frac{\partial g_1}{\partial y_1} \quad \frac{\partial g_1}{\partial y_2} \quad \frac{\partial g_1}{\partial y_3} \\ \frac{\partial g_2}{\partial y_1} \quad \frac{\partial g_2}{\partial y_2} \quad \frac{\partial g_2}{\partial y_3}}}$$

$Dg_2 \quad D_2 g_2 \quad D_3 g_2$

Example (wikipedia :)

Consider $f(t) = \int_0^{2\pi} \underbrace{e^{t\cos x}\cos(t\sin x)}_{\varphi(x,t)}\,dx$

a) Find $f'(t)$ $\Rightarrow D_2\varphi = \frac{\partial}{\partial t}\varphi = \cdots$

$$f'(t) = \int_0^{2\pi} \frac{\partial}{\partial t}\left(e^{t\cos x}\cos(t\sin x)\right)dx$$

$$= \int_0^{2\pi} \underbrace{e^{t\cos x}\left(\cos x\cos(t\sin x) - \sin(t\sin x)\sin x\right)}_{\frac{1}{t}\frac{\partial}{\partial x}e^{t\cos x}\sin(t\sin x)}\,dx$$

$$= \frac{1}{t}\int_0^{2\pi} \frac{\partial}{\partial x}\left(e^{t\cos x}\sin(t\sin x)\right)dx$$

$$= \frac{1}{t}\int_0^{2\pi} d\left(e^{t\cos x}\sin(t\sin x)\right)$$

$$= \frac{1}{t}\left(e^{t\cos x}\right)\sin(t\sin x)\Big|_{x=0}^{x=2\pi} = 0$$

(b) Evaluate $f(0)$ using the def'n of f

$\Rightarrow f(0) = 2\pi$

(c) Conclude that $\int_0^{2\pi} e^{\cos x}\cos(\sin x)\,dx = 2\pi$

Since $\frac{df}{dt} = 0$ then $0 = \int_0^1 \frac{df}{dt}\,dt = f(1) - f(0)$

$\Rightarrow F(1) = F(0) = 2\pi$

$\Rightarrow$ but $F(1)$ is $\int_0^{2\pi} e^{\cos x} \cos(\sin x)\, dx = 2\pi$.

